Regression: Transformations

Grinnell College

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Four Assumptions for the Linear Model

- Independence
  - Observations aren't correlated with each other
- Errors are normally distrubted with mean 0
  - Want a random scattering above and below the horizontal line at 0
- Homoskedasticity
  - The general spread of the residuals should be constant through the whole graph
  - Eg We don't want a megaphone shape pattern in the graph
- X and y's relationship is linear

General Plan

Can we do anything with violated assumptions?

- Yes, transformations!
- Logarithms
  - (FORMAT: Response Var.'s Scale - Explanatory Var.'s Scale)
  - Linear-Log
  - Log-Linear
  - Log-Log
- Power transformations
  - Square Root (generally of y)
  - Square (generally of x)
- Waaaay more that we can't get into

Independence

Sometimes data is temporally (or spatially) related

- Eg my chess rating over time
- Have techinques for this
  - Back shift operator: subtract this years ave. September temp from last year's ave. September temp
  - Moving average: our current prediction is the (weighted?) average of the last few time points
  - Auto-Regressive: Observations are correlated with their neighbors

Sometimes it's phsycially related

- 6 green onions grown in the same small flower pot
- Have techinques for this as well
  - Mixed Models (not in this class)
  - Side step the issue (average over the six green onions)

Non-Normality

Sometimes data just behaves differently

- CHECK FOR LURKING VARIABLE
  - A varible not yet looked at could drive strange behaviro
- I guess 3 heads out of four coin flips; I can only underguess by 1, overguess by 3 (not symmetric!)
  - Logistic Regression
- Discreet data with very few non-0 numbers
  - Discreet distribution over normal, eg Poisson Distribution
  - Zero-Inflated Regression Model
- It's just weird
  - Bootstrap (later) or simulations (also later)

Heteroskedasticity

A common violation of our assumptions is....heterskedasticity!

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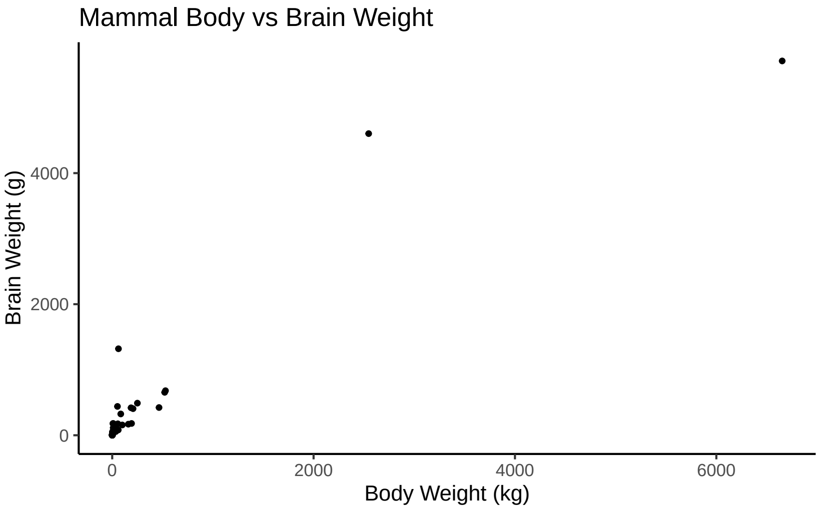
- It's common the st. dev. of your residuals changes depending on where you are at in the predicted vs residaul graph
- Usually (but not always) scientifically expected
- St. Dev. usually (but not always) balloons as the predictions increase

Is there a way to transform the data into a more "usable" form?

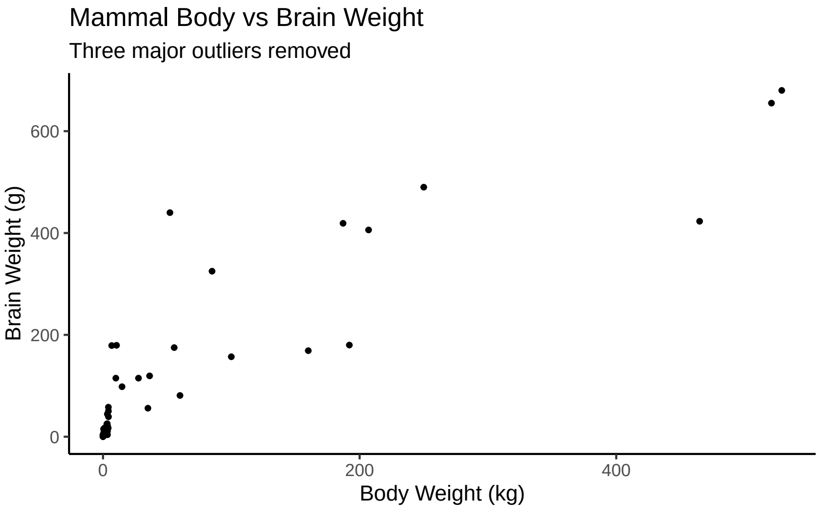
A BRIEF MESSAGE FROM OUR SPONSORS

Example



Mammal Body vs Brain Weight
Brain Weight (g)
4000
2000
0
0
2000
4000
6000
Body Weight (kg)

Example: Continued



Goal

We want data in a linear form, randomly scattered around a line with constant standard deviation (spread).

- We need a monotonic function that takes in (positive) numeric data
- and makes really large numbers not that large
- while keeping the small numbers relatively small
- and we want to be able to "back transform" (work backwards to get the original data).

Ideas?

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We want data in a linear form, randomly scattered around a line with constant standard deviation (spread).

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- and makes really large numbers not that large
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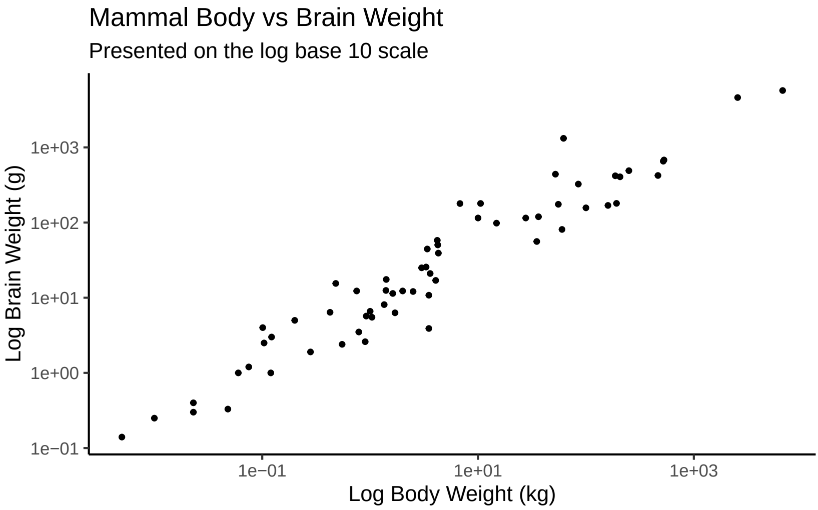
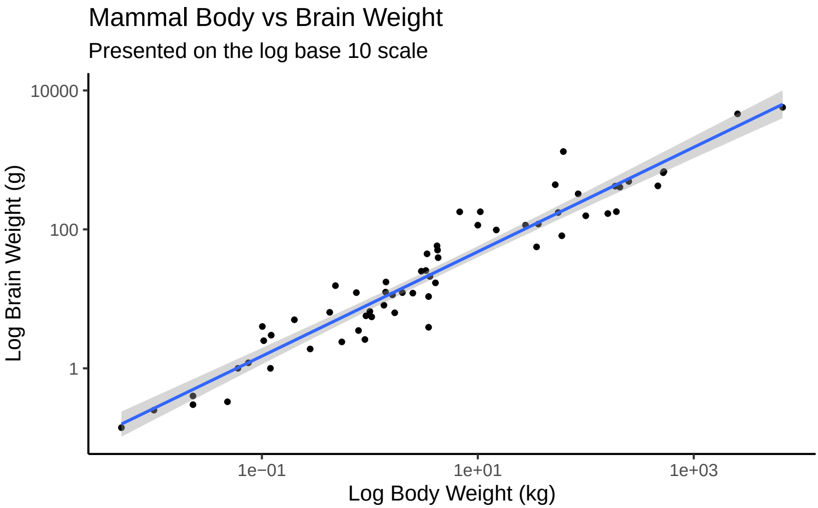
- and we want to be able to "back transform" (work backwards to get the original data).

Ideas? The logarithm does this, and so does the (positive) square root

Example: Continued Some More



Example: Predictions on Log-Log Scale



Back Transforming

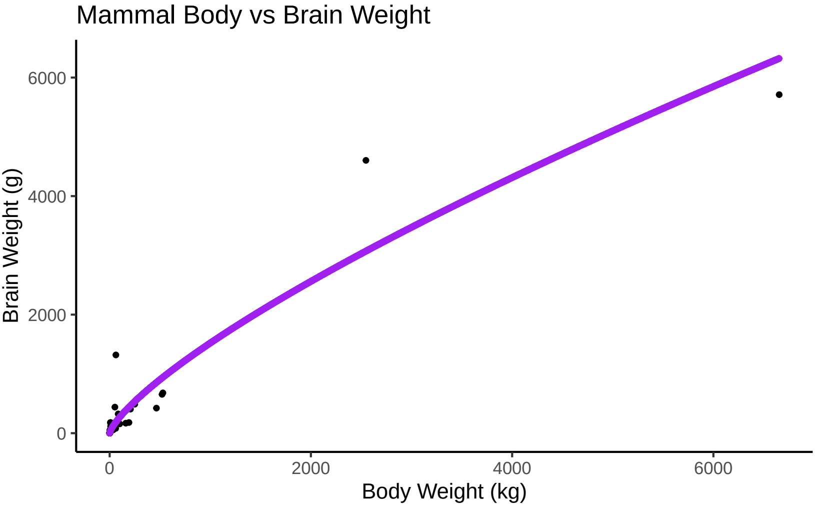
Back Transforming is applying a function to an already transformed variable to undo the transformation.

- Eg taking the square root of a squared variable
- Eg Undoing the log function by exponentiation
- Sounds more intimidating than it is

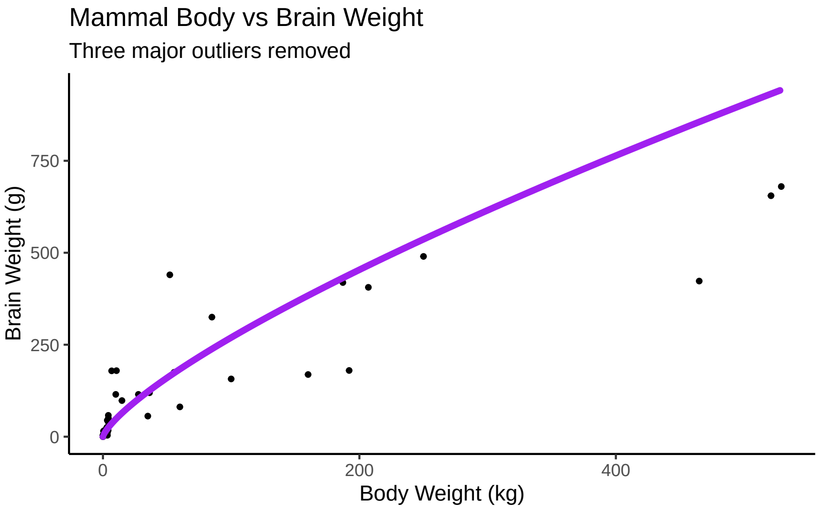
Usually back transformed values do well but not always

- Estimated spread (on the linear scale) balloons as you go up.
- Being off by .1 log units...
  - when the log value is low it's not bad (eg $10^{.5} = 3.16$ vs $10^{.6} = 3.98$)
  - when the log value is high is bad (eg $10^2 = 100$, $10^{2.1} = 126$)

Example: Predictions on Linear-Linear Scale



Example: Predictions on Linear-Linear Scale



So what just happened?

Instead of

$$y \ = \ \beta_0 \ + \ \beta_1 X \ + e$$

we fit

$$\log(y) \ = \ \beta_0 \ + \ \beta_1 \log(X) \ + e$$

Raw Value	Log_{10} Value
.2 $(=5^{-1})$	-.698
.5	-.301
1	0
5	.698
500	2.698

What's the catch?

Well.....

- We are modeling the log of the response, log(y), and not the response, y.
- Best-fit-line is guaranteed "best" only on the scale modeled
- And we are now fitting the median.....math is weird

Q: Wait, what??

Interpretations Background

Well.....we are modeling the log of the response, log(y), and not the response, y.

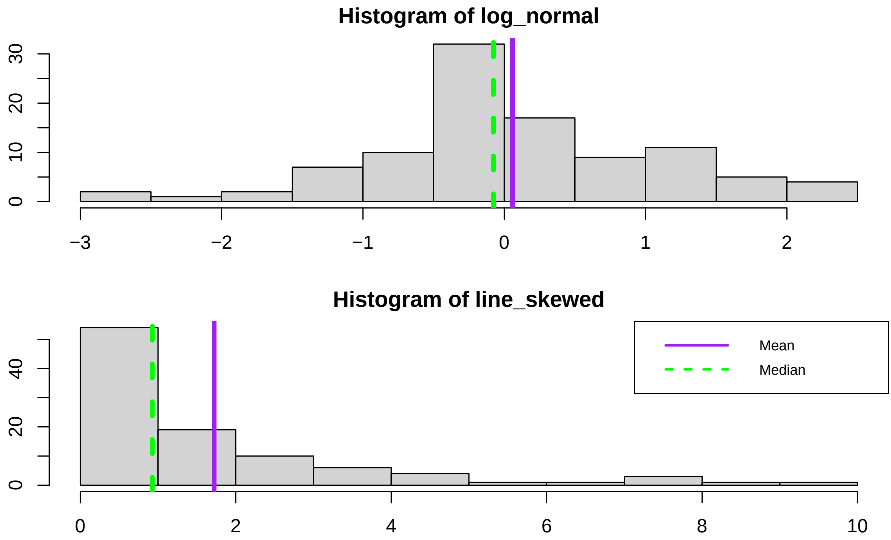
- We are assumign the errors are bell shaped on the log scale
  - The distribution will be skewed when we back transform
  - But the median remains the same

- Best-fit-line is guaranteed "best" only on the scale modeled

- And we are now fitting the median.....math is weird

Q: Wait, what??

Graphic Explanation



Estimated Log-Log Regression Equation

$$\hat{\log}(y) = \hat{\beta}_0 + \hat{\beta}_1 * \log(X)$$

$$\hat{y} = \exp(\hat{\beta}_0 + \hat{\beta}_1 * \log(X))$$

$$\hat{y} = \exp(\hat{\beta}_0) * \exp(\hat{\beta}_1 * \log(X))$$

Log-Log Model's Interpretation for $\hat{\beta}_0$

Two Interpretations:

BETTER:

When the log of the explanatory variable is 0, we expect the median response to be $\exp(\hat{\beta}_0)$

WORSE:

When the log of the explanatory variable is 0, we expect the (mean/median) of the log of the response to be $\hat{\beta}_0$

Slope Interpretation Primer

$$
\begin{aligned}
\hat{\log(y)} &= \hat{\beta}_0 + \hat{\beta}_1 * \log(X) \\
\hat{y} &= \exp(\hat{\beta}_0 + \hat{\beta}_1 * \log(X)) \\
\hat{y} &= \exp(\hat{\beta}_0) * \exp(\hat{\beta}_1 * \log(X))
\end{aligned}
$$

$$
\begin{aligned}
\hat{y_{new}} &= \exp(\hat{\beta}_0) * \exp(\hat{\beta}_1 * \log(1.10 * X)) \\
\hat{y_{new}} &= \exp(\hat{\beta}_0) * \exp(\hat{\beta}_1 * \log(X)) * \exp(\hat{\beta}_1 * \log(1.10)) \\
\hat{y_{new}} &= \exp(\hat{\beta}_0) * \exp(\hat{\beta}_1 * \log(X)) * 1.10^{(\hat{\beta}_1)} \\
\hat{y_{new}} &= \hat{y} * 1.10^{(\hat{\beta}_1)}
\end{aligned}
$$

Log-Log Model's Interpretation for $\hat{\beta}_1$

Two Interpretations

BETTER:

When the the explanatory variable increases by 10% we expect the median response to increase by a multiplicative factor of $1.1^{\hat{\beta}_1}$

WORSE:

When the log of the explanatory variable increases by 1, we expect the (mean/median) of the log of the response to increase by $\hat{\beta}_1$

Example Continued

$$\text{Predicted log(Brain Weight)} \; = \; 2.13 \; + \; .75 * \text{log(Body Weight)}$$

β_0: For a mammal with a log body weight of 0 (= the body weight of the animal is 1kg), then the predicted (median) weight of it's brain is $e^2.14 =$ 8.499g

β_1: If the body weight of the animal increases by 10%, then the predicted (median) brain weight increases by a multiplicative factor of $1.1^{.75} = 1.074$

Example Continued

Artic Fox has a body weight of 3.385kg, what is the predicted median weight for it's brain?

$$\text{Predicted log(Brain Weight)} = 2.13 + .75 * \text{log(Body Weight)}$$

Example Continued

Artic Fox has a body weight of 3.385kg, what is the predicted median weight for it's brain?

$$\text{Predicted log(Brain Weight)} \; = \; 2.13 \; + \; .75 * \log(3.385)$$

$$\text{Predicted log(Brain Weight)} \; = 3.044$$

$$\text{Predicted Brain Weight} \; = e^{(3.044)} \; = \; 20.999g$$

$$\text{Residual} \; = \; \text{Obs.} \; - \; \text{Predicted} \; = \; 44.50 \; - \; 20.999 \; = \; 23.501$$

Log-Linear

Estimate Log-Linear Regression Equation:

$$\widehat{\log(y)} = \hat{\beta}_0 + \hat{\beta}_1 * X$$

$$\hat{y} = \exp(\hat{\beta}_0 + \hat{\beta}_1 * X)$$

$$\hat{y} = \exp(\hat{\beta}_0) * \exp(\hat{\beta}_1 * X)$$

$\hat{\beta}_0$: ~~Interpretation is the same as for the log-log model~~
$\hat{\beta}_0$: If the explanatory variable is 0, then the median of the response is exp(), we believe

$\hat{\beta}_1$: For a one unit increase in X we expect the median response to increase by a mulitplicative factor of $\exp(\hat{\beta}_1)$

Linear-Log

Estimate Linear-Log Regression Equation:

$$\hat{y} \ = \ \hat{\beta}_0 \ + \hat{\beta}_1 * \log(X)$$

I don't know of any special interpretations to this one....useful if your x-axis is stretched out waaay too far

What's the point?

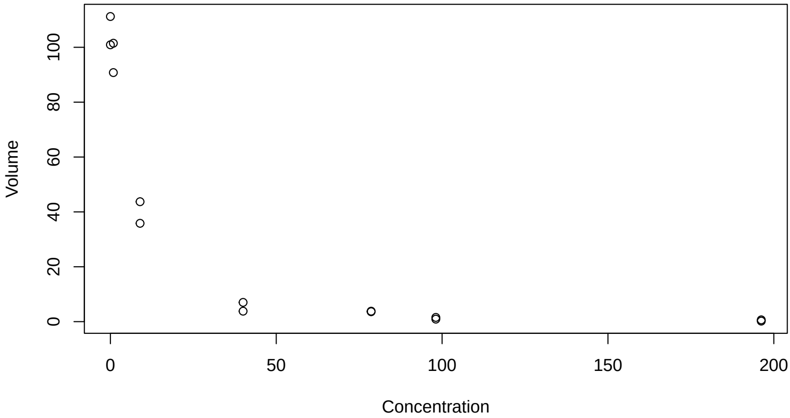
Our goal is to achieve a graph that is linear between whatever is on the x-axis and whatever is on the y-axis with a random scattering points above and below the line (4 assumptions)

- We have techniques help us a way to achieve these goals
- The log() transformation is popular to fix ballooning heteroskedasticity
  - Comes at a cost
  - No "best" properties on the human scale
  - The spread of our predictions balloons
  - Comments can be restricted to medians, not means
- Transformations in general can help us with this goal

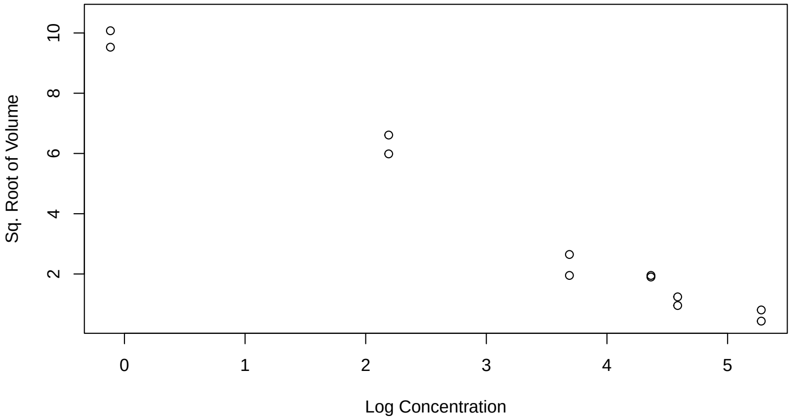
Other Transformations

- Not uncommon for X to be replaced with X^2
- The response can be raised to a power
  - Often square root or cubic root
  - Box-Cox Transformation gives a formula to optimize what power to raise the response to
  - Useful if the units give an indication (see next slide)
- Generalized linear models (GLM's) is a way to deal with the non-normal, quirky data via a different distribution
  - Eg Logistic and Poisson regression
  - Eg Exponential family
- Finally, non-parametric methods such as ranking the data
  - Calling Spearman's Correlation....

Algae Blooms



Algae Blooms: Transformed



Next Time

How do we make a linear model when we have categories for our explanatory variable?