Z-tests

Grinnell College

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Sampling Distrbutions for...

- Numeric data with known variance
  - Z-test
  - Confidence intervals
- Numeric data with unkown variance
  - t-test
  - Confidence intervals
- Difference of two numeric variables (with unkown variance) and ANOVA
  - t-tests and ANOVA
  - Confidence intervals

Next Unit

- Categorical data with proportions
  - 1 proportion (test + confdience interval)
  - 2 proportions (test + confidence interrvals; super brief to say I actually did show it to you)
  - ^{2}-test for tables
- Regression
  - t-tests and conf. int. for coefficients
  - Multiple Regression
- Logistic Regression

Starting Point

We talked about how statistics calculated from a random sample will have a long term behavior known as a sampling distribution.

The sample mean $\bar{x}$ actually has a distribution, it's...

$$\bar{X} \sim N(\mu, \sigma^2/n) \tag{1}$$

- only true if the assumptions hold
- μ is the population mean
- σ^2 is the population variance
  - For this slide deck we will act like we know this
- n is our sample size

Assumptions

First we have to check if the assumptions are correct....

- The observations are randomly sampled
- The observations are independent and come from the same population
  - "independent and identically distributed" = "iid"
- The population is normally distrubted or the sample size is large
  - How can we tell if the population is normally distributed?

Assumptions

First we have to check if the assumptions are correct....

- The observations are independent and come from the same population
  - "independent and identically distributed"
- The population is normally distributed OR the sample size is large
  - How can we tell if the population is normally distributed? We use the sample distribution which should look like the population distribution
  - The sample size being large enough kicks in the central limit theorem (next slide)

If both main bullet points are checked off we can use the normal distribution to approximate the sampling distribution

Central Limit Theorem

The **central limit theorem** (CLT) says the the long term behavior of an average will be normally distributed if the sample size (eg people in a single study) is sufficently large

- Large sample size needed (called an asymptotic result)
- Makes the sampling distribution normal
- There is a requirement that the population distribution actually *has* a mean and variance
  - Never(?) questioned in practice
  - Take much higher level math classes for more details
  - Eg Cauchy distribution doesn't have a mean

Central Limit Theorem

What is a large sample size?

- 30 is the classic response (half joke anymore?)
- Depends on how not-normal the population distribution is
  - Again, we use the sample distribution in practice
  - If symmetric without outliers N $\approx$ 20
  - Skew or some outliers N $\approx$ 50
  - Really really weirdly shaped N $\approx$ 100
- I personally have great faith in the CLT

Once you hit a sample size of 100 or more the CLT will do well for pretty much anything

Regression Flashback

The old assumptions....

1. X, Y's relationship is correctly identified (a linear model)
2. The observations are independent and identically distributed
3. The errors are normally distributed with mean 0

The third point is relevant. For z-tests (right now) we focus on the populations directly being normal.

This will become relevant next unit.

Regression Flashback

The old assumptions....

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What about random?

I kicked this till after the midterm but it's traditonally taught as an assumption (very fair!).

The reason we want randomly collected data is that it gets us our iid assumption. Without random collection/assignment we cannot justify independence and identically distributed (usually)

Example 1

Over the course of a few years in the mid-2000's a collection of body measurements were taken on colonies of penguins, randomly selected, on some islands off of Antarctica. Find the sampling distribution for mean flipper length

Example 1: Assumptions

Are the observations random?

- ?

Are the observations independent and identically distributed?

Is the population normally distributed or is the sample size large?

Example 1: Assumptions

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- Yes, it says so

Are the observations independent and identically distributed?

- ?

Is the population normally distributed or is the sample size large?

Ex: Penguin Assumptions

Are the observations random?

- Yes, it says so

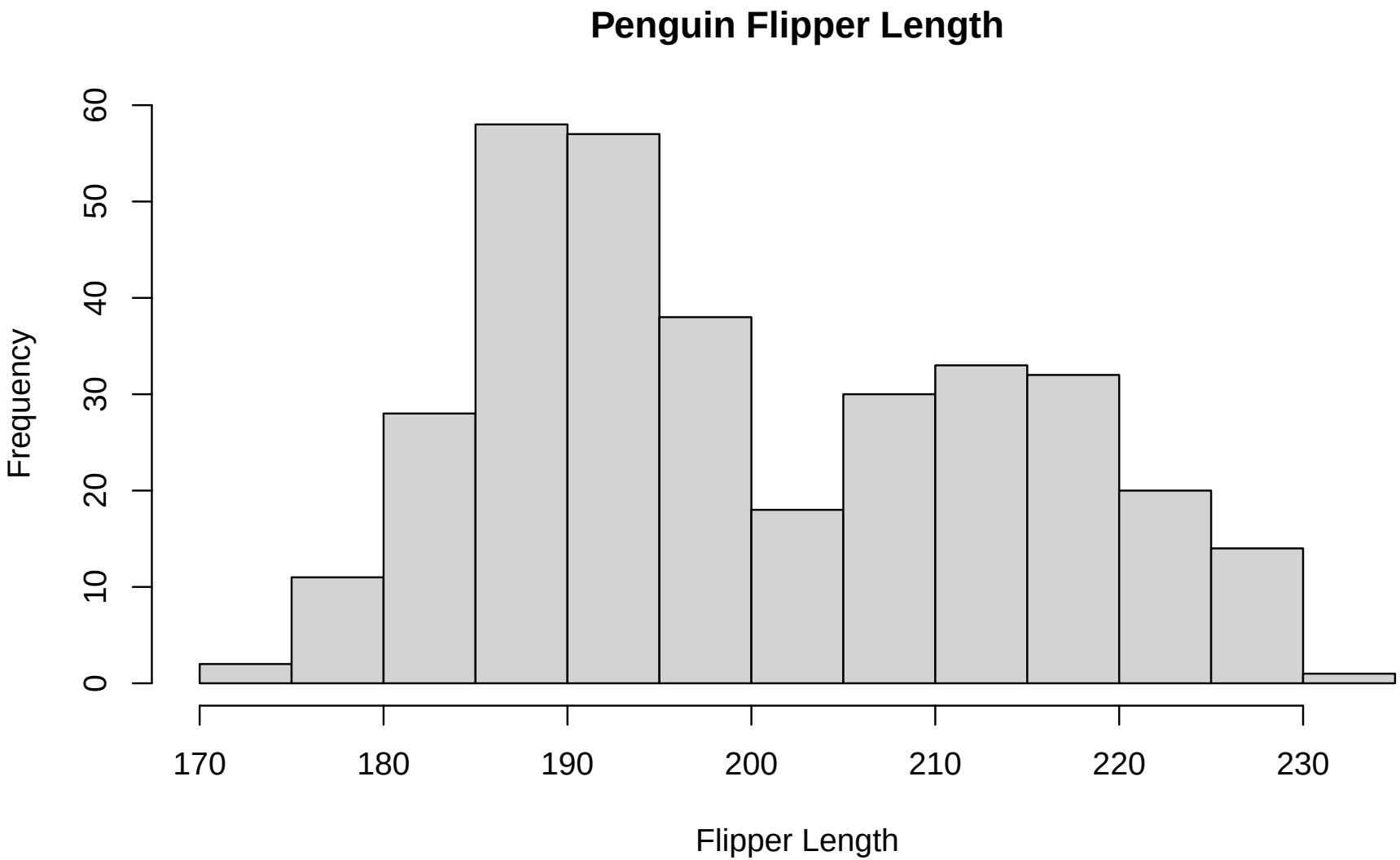
Are the observations independent and identically distributed?

- No! We already have explored that different subpopulations of penguins exist (first midterm!)
- Our results are going to be untenable because of that
  - We are ignoring an important **covariate** (another variable that is important to the model)
  - And instead it becomes a **lurking variable** (a covariate that is ignored in the analysis)

Is the population normally distributed or is the sample size large?

- The population.....?

Ex: Penguin Plot



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Is the population normally distributed or is the sample size large?

- The population is not normally distributed (because the sample isn't)
- The sample size is....

Ex: Penguin Assumptions

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- The population is not normally distributed (because the sample isn't)
- The sample size is....large with over 300 observations

Sampling Dsitrubtion for a Single Mean

The **sampling distribution for a single mean** will be....

$$\bar{X} \sim N(\mu, \sigma^2/n)$$

where...

- μ is our population's mean
  - Unknown
  - We can estimate it with our sampel
  - Or we can hypothesize it's value
- σ^2 is our variance
  - Population's variance
  - Almost always unknown and must be estimated
  - For this slidedeck assume we know the true variance
- n is our sample size

Ex: Penguin Sampling Distribution

The resulting sampling distribution then is N(μ, σ^2/n) which is

$$\bar{X} \sim N(\mu, 180/344)$$

where

- μ is the mean of our population
  - We can estimate it with out sample mean (200)
  - We can hypothesize it's value
- 180 is our (pop) variance
  - Assumed known before we ever started talking about penguins
  - Very unrealistic short of an ornithologist savant
- 344 is our sample size

Let's review hypothesis tests real fast and then we can try some

Z-Test

1. State our two hypothesis
2. ~~Define a thresh hold/critical value~~
3. Visualize your data (already done)
4. Check assumptions (already done)
   - The sample was randomly collected/drawn
   - Independence from the same population
   - Normal or Large N
5. State your sampling distribution
6. Test Statistic
   - Standardized value we observed
7. Probability we see that test statistic (or more extreme)
   - Calculate the probability from our (normal) sampling dist.
8. Decision is made
   - profit

Step 1: Hypothesis Statement

The **null hypothesis** is the hypothesis you are wanting to show is most likely wrong

- Denoted as H_0
- Will use $=$, $\geq$, or $\leq$

The **alternative hypothesis** is the other possible state of the world that is not the null hypothesis.

- Defined as what the null hypothesis isn't
- Will use $>$, $<$, or $\neq$
- Usually this is what you want to show

For means, we will use the (unknown) parameter μ

Ex: Penguin Z-test

My uncle: *The average penguin flipper length? I don't know Vinny I'd guess a foot long maybe*

Let's try to disprove my uncle's (unwilling) claim that a penguins flipper is about a foot long (305 mm).

Step 1 is to write out our hypothesis

Ex: Hypothesis Statements

You must list exactly two hypothesis statements, the null and the alternative.

$$H_0 : \mu_{flipper} = 305$$
$$H_A : \mu_{flipper} \neq 305$$

Here we are trying to show there is a difference; that RJ's claim they the flippers are a foot long is just wrong.

Sampling Distribution's Mean for a Hypothesis Test

Critical: During a hypothesis test we assume we *know* the population mean (via H_0)

- We assume the sampling distribution is centered at the population's mean
- We make some claim about the pop's mean in H_0
- So plug in the hypothesized value from H_0 into the sampling distriubtion

When we judge whether the results we got are unlikely or not; we are judging in effect how believable H_0 is.

Ex: Penguin's Sampling Dist for 1 Mean HT

Initially we had...

$$\bar{X} \sim N(\mu, 180/344)$$

but we still need to replace μ with out hypothesized value so...

- $H_0 : \mu_{flipper} = 305$
- $180/334 = .54$

$$\bar{X} \sim N(305, .54)$$

Ex: Penguin's Test Statistic

For a Z-test, the test statistic is the standardized observation so....

- The sample mean was 200
- Hypothesized mean is 305
- And the standard deviation for our sampling distribution is $(\frac{\sigma^2}{n})^{1/2} = (180/334)^{1/2} = .7341$
  - This is called the **standard error**; it's the standard deviation of the sampling distribution
  - Personally I think it having it's own name is kind of.....not wise
  - Standard deviation (square root of a distribution's variance) vs standard error is a common question to split the wheat from the chaffe

$$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{200 - 305}{.7341} = 143$$

Probability we'd see this (ie p-value)

Let T be our test statistic from the previous step and Z represent a standard normal variable. Three possible situations based on the alternative hypothesis:

- P(Z $<$ T)
  - H_A: $\mu <$ (SOME NUMBER)
  - Ie take the probability to the left
- P(Z $>$ T)
  - H_A: $\mu >$ (SOME NUMBER)
  - Ie take the probability to the right
- 2 * P(Z $>$ $|T|$)
  - H_A: $\mu \neq$ (SOME NUMBER)
  - This sums the probability we'd be lower than -$|T|$ or greater than $|T|$
  - We multiply but 2 because we use symmetry to make the equation easier

Ex: Penguin's Weirdness (ie P-value)

NOTE: Because we are using $\neq$ in our alternative hypothesis we need....

$$2 * P(Z > \ |T|) = 2 * P(Z > \ |143|)$$

Which is equivalent to...

$$P(Z < -143 \cup Z > 143) = P(Z < -143) + P(Z > 143)$$

2 * pnorm(143, mean = 0, sd = 1, lower.tail = FALSE) ≈ 0

The probability that we'd see an average of 200mm given the real mean is 305mm is below R's reporting ability to calculate (eg R just says "0")

Decision

Based on our p-value, we have a few decisions we can make but they generally follow the this format....

"We have (BLANK) evidence to suggest the true (STATISTIC) is (H_A)"

- BLANK is very strong, strong, moderate or weak (more next slide)
- STATISTIC is the statistic we are interested in
  - Usually a mean or proportion in this class
- Note that the decision is wrote in the terms of evidence of H_A
  - Kind of weird if you think about it

Decision: How strong is strong?

Different fields in different context want differing level of evidence. For example....

- A business owner might want to know if they should go with brand X or Y; they may just want bare evidence one is better
- FDA wants evidence a med works before it's released to the public
- Physicits run an experiment and it weakly suggests gravity doesn't exist

Despite this, broadly we still have classifications based on the p-value size.

Decision: How strong is strong?

P-Value	Level of Evidence
$> .10$	Little to No Evidence
.10 - .05	Weak Evidence
.05-.01	Moderate Evidence
.01-.001	Strong Evidence
$< .001$	Very Strong Evidence

Decision: How strong is strong?

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$> .10$	Little to No Evidence
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Recall: p-value observed was roughly 0 (extremely small)

Decision: How strong is strong?

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$> .10$	Little to No Evidence
.10 - .05	Weak Evidence
.05-.01	Moderate Evidence
.01-.001	Strong Evidence
$< .001$	Very Strong Evidence

Recall: p-value observed was roughly 0 (extremely small)

Decision: We have very strong evidence that the mean flipper length for penguins is different than 305.

Ex: Decision

"We have (BLANK) evidence to suggest the true (STATISTIC) is (H_A)"

becomes

"We have (very strong) evidence that the (mean flipper length for penguins) is (different than 305.)"

- BLANK = very strong
- STATISTIC = mean flipper length for penguins
- $H_A = \mu \neq 305$

Another Example

Lumber from cherry trees can be quite beautiful running from a pinkish brown to a reddish color. A common goal for woodworkers is to get long planks which is a function of the height of the tree. I'm willing to claim that the mean of cherry trees is shorter than 75 feet tall.

31 black cherry trees were randomly selected from a forest and measured with a mean height of 73.

It's known that the population's variance is 45 ($= \sigma^2$)

Steps

1. State our two hypothesis
2. ~~Define a thresh hold/critical value~~
3. Visualize your data
4. Check assumptions
   - Independence from the same population
   - Normal or Large N
   - Sample was randomly collected/drawn
5. State your sampling distribution
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Hypothesis

Note that our hypothesized value is from my claim and not the sample (common mistake)

$$H_0 : \mu \geq 75$$
$$H_A : \mu < 75$$

Also note that our alternative hypothesis is written for what we want to show

Visualize


Histogram of trees$Height
Frequency
0
2
4
6
8
10
60
65
70
75
80
85
90
trees$Height

Assumptions

- Random?
-
-

Assumptions

- Random? says so...
- Independent?
-

Assumptions

- Random? says so....
- Independent? Probably but the trees could be competing against each other but that seems unlikely
- Normally distributed population or large N
  - Normal Pop?

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  - Large N?

Assumptions

- Random? says so....
- Independent? Probably but the trees could be competing against each other but that seems unlikely
- Normally distributed population or large N
  - Normal Pop? The sample is normal so we think so
  - Large N? Yes for symmetric, hill shaped data 31 is plenty although it's a moot point

Sampling Distribution

Given that we passed the assumptions we believe the sampling distribution for the sample mean will be...

N(75, 45/31)

- 75 is our hypothesized value from H_0
- 45 is our population's (known) variance
- 31 is our sample size

Test Statistic

$$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{73 - 75}{1.45} = -1.377$$

P-value and Decision

P(Z $<$ -1.377) is .087

We have weak evidence that the mean height of black cherry trees is shorter than 75 feet

Misconception

If our p-value is large we do NOT claim we have evidence in favor of H_0

Why?

Misconception

If our p-value is large we do NOT claim we have evidence in favor of H_0

Why?

It can be wrong but our n is too small to detect a difference